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NOTE ON THE CALCULATION OF BOUNDARY LAYERS

By L. Prandtl

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NOTE ON THE CALCULATION OF BOUNDARY LAYERS*

By L. Prandtl

SUMMARY

The properties of the solutions of the hydrodynamic equations of viscous fluid by "boundary-layer omission" are discussed. A method is indicated for the numerical determination of the solution for a known initial profile $u(x_0, y)$ and pressure distribution $p(x)$ within the region.

1. INTRODUCTION

In flows with small viscosity (large Reynolds number) the field of viscosity is frequently restricted to a small zone in the neighborhood of the body surface. To facilitate the calculation in this case, the following assumptions may be made:

1) In the frictional zone, termed "boundary layer," only the strongest terms of the viscosity force $\mu\Delta\underline{u}$ are taken into account (thus, for instance, on the x component of this force $\mu\Delta u$ - when the principal direction of flow parallel to the wall is chosen as x direction, and that at right angle to it as y direction - $\mu \frac{\partial^2 u}{\partial x^2}$ and $\mu \frac{\partial^2 u}{\partial z^2}$ are neglected relative to $\mu \frac{\partial^2 u}{\partial y^2}$.

2) Outside of the boundary layer the flow is treated as frictionless and, because of the small extent of the boundary layer at right angle to the wall, the pressure in this layer is equated to that which the related frictionless flow

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*"Zur Berechnung der Grenzschichten." Zeitschrift für angewandte Mathematik und Mechanik, vol. 18, no. 1, Feb. 1938, pp. 77-82. (Special Reprint)

would furnish on the wall.* The trifling motion of the pressure with the wall distance within the boundary layer is disregarded, hence the pressure considered as a given function of the coordinates (x and z).

For an average speed u_0 of the outer flow and a flow length l, the calculation gives for the "boundary layer thickness" δ the order of magnitude

$l\sqrt{\frac{\nu}{u_0 l}} = l/\sqrt{R}$ ($\nu = \mu/\rho$ = kinematic viscosity).

For the two-dimensional case with x = arc length measured along the wall in flow direction and y = vertical distance on the wall, the equations read:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{dp}{dx} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2)$$

Of general interest is the stationary case $\left(\frac{\partial u}{\partial t} = 0\right)$.

about which much as been written (see reference 1), although there still is no satisfactory solution for one important aspect; namely, the further development of a given velocity profile by given pressure distribution. This problem forms the subject of the present discussion. In mathematical language it is expressed as follows:

Given: the velocity profile $u(x_0,y)$ for the <u>initial section</u> $x = x_0$, and, in addition,

$\frac{1}{\rho}\frac{dp}{dx} = \nu f(x)$ for $x \geq x_0$;

To find: $u(x,y)$ for $x > x_0$.

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*More precisely, the frictionless flow should be taken on a body thickened up by the displacement thickness

$$\delta^* = \frac{1}{u_1}\int_0^{\delta} (u_1 - u)\,dy.$$

2. PROPERTIES OF THE STATIONARY BOUNDARY LAYER FLOW

The known solutions contribute nothing to this problem because they contain merely expansions in powers of x from an initial section, wherein $u = 0$ or $u = \text{const}$ (stagnation-point flow and flow past an edge with zero edge angle). On the other hand reference may be made to the fact that - discounting the case of break-away of flow - the existence of the solution is physically evident: a boundary layer profile created during any preceding period must, so long as no separation occurs, continue to develop in some way or other. The mathematical answer to this question, which I already held at the first mathematical congress at Heidelberg in 1904, is as follows:*

By the use of equation (2), $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$ in equation (1) may be rewritten as: $v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y} = -u^2\frac{\partial}{\partial y}\left(\frac{v}{u}\right)$; hence,

$$\frac{v}{u} = \nu \int_{y_0}^{y} \frac{1}{u^2}\left(f(x) - \frac{\partial^2 u}{\partial y^2}\right) dy + \varphi(x) \tag{3}$$

$\frac{v}{u}$ is the slope of the streamline toward the x axis.

If the axis itself is the wall, then $\frac{v}{u} = 0$ for $y = y_0 = 0$, i.e., $\varphi(x) = 0$. As a secondary product, it is seen that in equation (3) the case of a (slightly) wavy wall is also included, which is different from the x axis. It is, then,

$$\frac{dy_0}{dx} = \left(\frac{v}{u}\right)_{y=y_0} = \varphi(x)$$

*In the text of the speech at the III Int. Math. Cong., Leipzig, 1905, and in the new edition by Prandtl-Betz entitled: Vier Abhandlungen zur Hydrodynamik und Aerodynamik, Göttingen, 1927, p. 487, it reads: if, as usual, dp/dx is given, and, in addition, the curve u for the initial section, every such problem can be numerically dealt with, by obtaining through quadratures from each u the relative $\partial u/\partial x$; in this manner it is always possible to move a step forward in the x direction with the aid of one of the known approximate methods. A difficulty exists of course in different singularities occurring on the fixed boundary.

which leads to

$$y_0(x) = \int_{x_0}^{x} \varphi(x)\, dx$$

as equation for the wall. This secondary result can also be expressed so that, if $u(x,y)$ is a solution of the system (1), (2), $u(x,y - y_0(x))$ itself is a solution. Naturally, this can also be proved direct from (1), (2).

The significance of equation (3) for our problem, however, rests on the fact that $\partial v/\partial y$ and hence $\partial u/\partial x$ can be obtained by means of equation (2). For $\varphi(x) = 0$, it is

$$\frac{\partial u}{\partial x} = \nu \frac{\partial}{\partial y}\left[u \int \frac{1}{u^2}\left(-f(x) + \frac{\partial^2 u}{\partial y^2}\right) dy\right] \tag{4}$$

The right-hand side of this equation merely contains differentiation and integration with respect to y, and is therefore amenable to complete calculation for an initial profile $u = u(x_0,y)$, for which it yields $\partial u/\partial x$ and with it the possibility of continuing with respect to x, provided that $\frac{\partial u}{\partial x}$ is restricted. It gives a new u for $x_0 + \Delta x$:

$$u(x_0 + \Delta x,\, y) = u(x_0,y) + \frac{\partial u}{\partial x}(x_0,y)\, \Delta x$$

The method can be repeated for this new profile and yields a $u(x_0 + 2\,\Delta\, x,\, y)$ etc., thus proving the existence of the solution for all zones wherein u remains positive. On the other hand, it is readily apparent that the method fails if u passes anywhere through zero, where the integrand becomes quadratically infinite. Now, however, $u = 0$ along the wall, that is, for $\varphi(x) = 0$ along axis x, because of the adherence of the fluid to the wall. The condition is ameliorated because v becomes at the same time quadratically zero.* Now, a glance at equation (3)

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*With u, $\frac{\partial u}{\partial x} = 0$ also, for $y = 0$; but $v = -\int_0^y \frac{\partial u}{\partial x}\, dy$ because of equation (2).

indicates that the function $\frac{v}{u}$ is regular only when the integrand is everywhere finite. This stipulates that, on the wall, the expression $f(x) - \frac{\partial^2 u}{\partial y^2}$ quadratically disappears, i.e., that $\left(\frac{\partial^2 u}{\partial y^2}\right)_{x=x_0} = f(x)$ and $\left(\frac{\partial^3 u}{\partial y^3}\right)_{x=x_0} = 0.$

The evaluation of the "lim" then gives $-\frac{1}{2}\left(\frac{\partial^4 u}{\partial y^4}\Big/\left(\frac{\partial u}{\partial y}\right)^2\right)_{x=x_0}$ for which equation (4) gives the limiting value of $\frac{1}{y}\left(\frac{\partial u}{\partial x}\right)$ for $x = x_0$, $y = 0$ at $+\nu\left(\frac{\partial^4 u}{\partial y^4}\Big/\frac{\partial u}{\partial y}\right)_{x=x_0}$. And this means that

1) The second and the third differential quotients of the initial profile for $y = 0$ cannot be arbitrary, but rather must be $= f(x_0)$ and 0.

2) The fourth differential quotient at $y = 0$ is decisive for the further development of the profile.

For a contemplated second step, therefore, the initial profile $u(x_0,y) + \left(\frac{\partial u}{\partial x}\right)_{x=x_0} \Delta x$ already contains $\frac{\partial^4 u}{\partial y^4}$. The new "lim" consideration shows that the fifth and the sixth differential quotients are again bound and the success of the second step in wall proximity depends on the seventh differential quotient of the initial profile at $y = 0$! Now it must be considered how, for instance, the seventh differential quotient is to be determined on a tabulated function, and at the one end of the table of functions at that. And all that is accomplished here are but two short steps.

On top of that the bonds! It is seen from equation (3) or (4) that a violation of the bonds leads to a singular behavior of the solution at points $x = x_0$ and $y = 0$.

Two questions arise:

1) How is the vicious action of the equation system

(1), (2) in the vicinity of places with $u = 0$ to be understood?

2) Of what type are the singularities which occur on a regular, but the bonds violating, initial profile?

3. PROCEDURE OF SOLUTION AT PLACES WITH $u \lesseqgtr 0$

At this instance it is first to be stated that the Navier-Stokes equations are closely related to the biharmonic equations ($\Delta\Delta F = 0$), whence the solutions within the zone are always analytic. The introduction of the boundary layer omissions, however, represents a character-changing intervention. The structure of equation (3) itself indicates the presence of characteristics (namely, the straight lines parallel to axis y) along which discontinuities occur, if $f(x)$ or $\varphi(x)$ have discontinuities. This points therefore to a parabolic character. By a contact transformation by means of which the stream function ψ is introduced as an independent variable,* the case becomes clearer. Posting

$u = \frac{\partial\psi}{\partial y}$ and $v = \frac{\partial\psi}{\partial x}$, equation (2) is identically satisfied. Writing the new coordinates ξ, η in place of x,y in equation (1), and posting $\xi = x$, $\eta = \psi$ gives

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*In the spring of 1914 I was occupied with the flow through narrow channels and found, when applying the usual boundary layer equations, some difficulty in formulating the limiting conditions $u = v = 0$ on the second edge. I therefore attempted to introduce the stream function as an independent variable, since the two edges now simply read $\psi = 0$ and $\psi = Q = \text{const}$. Equation (5) was subsequently rediscovered by Von Mises and reported at the Kissinger meeting of the Society for Applied Mathematics and Mechanics in the fall of 1927 (Z.f.a.M.M., vol. 7, 1927, p. 429). Since I failed at that time to publish my own calculation method, Von Mises has therefore the usual priority. For the rest, the reader is referred to my remarks incidental to Von Mises' report in this periodical, vol. 8, 1928, pp. 249 and, particulary section 2, 250.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} - v\frac{\partial u}{\partial \psi}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = 0 + u\frac{\partial u}{\partial \psi}$$

$g = p + \frac{\rho}{2}u^2$

whence from equation (1) follows

$\frac{\partial g}{\partial x} = \frac{\partial p}{\partial x} + \rho u \frac{\partial u}{\partial x}$

$$u\frac{\partial u}{\partial \xi} + \frac{1}{\rho}\frac{dp}{d\xi} = \nu u \frac{\partial}{\partial \psi}\left(u\frac{\partial u}{\partial \psi}\right)$$

$\frac{\partial g}{\partial \psi} = \overset{0}{\frac{\partial p}{\partial \psi}} + \rho u \frac{\partial u}{\partial \psi}$

The "total pressure" $g = p + \frac{\rho u^2}{2}$ (the vanishing amount $\frac{\rho v^2}{2}$ being disregarded) introduced herein and x substituting for ξ, and posting $\nu\rho = \mu$ gives

$u\frac{\partial u}{\partial x} + \frac{1}{\rho}\left(\frac{\partial g}{\partial x} - \rho u\frac{\partial u}{\partial x}\right) = \nu u \frac{\partial}{\partial \psi}\left(u\frac{1}{\rho u}\frac{\partial g}{\partial \psi}\right)$

$$\frac{\partial g}{\partial x} = \mu u \frac{\partial^2 g}{\partial \psi^2} \quad = \nu u \frac{\partial^2 g}{\partial \psi^2} \qquad (5)$$

where $u = \sqrt{2(g - p(x))/\rho}$. Equation (5) is then a differential equation which is closely related to that of heat conduction (ψ in place of bar length and x instead of time), and of the parabolic type of which is known that on it a propitious direction with analytical disappearance of temperature differences, etc. (increasing time periods) exist, but at the same time also a deleterious direction with occurrence of singularities (extrapolation to states which with respect to time are before the given initial distribution).* On equation (5), similar to the problem of heat conduction, the development with respect to increasing values of x is propitious so long as μ is positive. But where μ is negative, as behind the break-away point, the deleterious state begins. Physically this means that the wall-adjacent fluid particles behind the break-away point move from greater toward smaller x, whence their velocity distribution depends upon the phenomena's taking place at these greater x values. Attempts to define

*"How should the distribution have looked before a minute (or hour) in order that the momentary nonuniform temperature distribution originates from it" (evidently frequently an unsolvable problem!).

then from the states at smaller x is, at best on a par with guessing games. In other words: the correct subsequent development of the flow conforming to observations aft of the "break-away point" is predicated on the careful selection of our function $f(x)$ to fit the desired process.

As concerns the behavior of the boundary layer on the wall itself in the "normal region" $(u > 0 \quad \text{for} \quad y > 0)$, the analytical continuation of u in the region of negative y leads to negative values of u, if, as stipulated, $\frac{\partial u}{\partial y}$ is positive for $y = 0$. The tendency of the solution toward singularities along the wall is thus explained by the fact that the analytical continuation through the wall leads into a "detrimental zone." To the extent that power series with respect to y are available for separate solutions, the extrapolation can be achieved. It shows in the zone of negative y a continuous ascent of the function values, such as $y\,e^{\alpha y^2}$ or similar functions.

4. SOLUTION BY VIOLATION OF THE "BONDS"

For a long time I unsuccessfully suggested the problem cited at the end of section 2 as a thesis for a degree of doctor until in 1929 Dr. S. Goldstein, Cambridge (England) as scientific guest of the Institute undertook this problem and explored it thoroughly. The singularities induced by an initial profile violating "bonds" are of algebraic nature. It is found that every initial profile developable in power series for y furnishes for a given dp/dx as whole function a solution which appears as power series for $\sqrt[3]{x - x_0}$; the power coefficients themselves being functions of $\frac{y}{\sqrt[3]{x - x_0}}$, defined by differential equations of the third order. (Details may be found in Goldstein's report entitled "Concerning some solutions of the boundary layer equations in hydrodynamics," Proc. Cambr. Philos. Soc., vol. 26, I, 1930, p. 1.) For the problem in hand this method of solution does not appear to be immediately applicable. And even if it could be continued, the gain would not be much because 3n functions would have to be formulated in order to reach the nth power of $x - x_0$ and that involves a further increase in the amount of paper work with every power of $\sqrt[3]{x - x_0}$. Goldstein figured with three terms and even then required very complicated developments in or-

der to establish the connection with the given function values. He thus reached only the linear term, whence the step Δx would have to remain very small so as not to introduce inadmissible errors. The new profile would then have to be represented again as power series of y in order to prepare the next step. The whole process would therefore become extremely laborious and hardly profitable.

There is therefore the greatest interest for continuation with integral powers of x. And the subsequent argument makes the prospects for a practical way toward this end, appear hopeful. If it involved the continuation of a solution already achieved by a previous method for a completely regular course of the pressure gradient, the collective bonds should, by rigorously effected calculation for the preceding section already be satisfied in the end profile which is to serve as initial profile of the new section. They are not exactly such in reality, for the series expansion in x is either stopped after a comparatively low term or else numerical inaccuracies have occurred. So, ordinarily, it will merely become a matter of reestablishing the bonding conditions by very slight changes in the given initial profile, after which the continuation in integral powers of x may proceed.

The question, how eventual errors made by this shifting of the profile affect the subsequent steps, is in order. this will involve mostly such departures from the unknown exact form, now winding positive, then negative, around the exact curve. According to the theory of heat conduction, such short-wave fluctuations in the temperature distribution die out relatively quickly. Hence only the proximity of the value $u = 0$ needs special study, and for this very reason I suggested the case of a small disturbance ϵ of oscillating course superimposed on a velocity profile of the simple form $u = a y$. Stipulating, for reasons of simplification, that ϵ shall be of the form $\epsilon = f(x) \varphi(y)$, it follows from systems (1),(2), after linearization, that it is necessary to put $f(x) = e^{-\alpha x}$. For $\varphi(y)$ the differential equation then gives

$$a \alpha y \varphi'(y) + \nu\varphi'''(y) = 0$$

which is solved by

$$\varphi(y) = A \int_C^y \sqrt{y}\, J_{-1/3}\left(\frac{2}{3}\sqrt{\frac{a\alpha}{\nu}}\, y^{3/2}\right) dy$$

$$+ B \int_0^y \sqrt{y}\, J_{+1/3}\left(\frac{2}{3}\sqrt{\frac{a\alpha}{\nu}}\, y^{3/2}\right) dy + C$$

(This solution was supplied by my co-worker, Dr. H. Görtler. He also adduced the proof that a solution for $f = e^{+\alpha x}$ satisfying one of the limiting conditions does not exist.) Of these expressions, however, only that with A satisfies the limiting conditions $\varphi(y) = 0$ and $\varphi''(y) = 0$; hence the physical solution in question is restricted to this term. The power expansion of y gives a term with y, another with y^4, one with y^7, etc., as corresponds to the bonds for the case $f(x) = 0$. As for the rest, the function oscillates with decreasing amplitude and decreasing wave length for increasing y.* This behavior of the solution may therefore be cited as proof that, by any small errors of $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ in the velocity profile employed for continuing the solution, no aftereffects exceeding the amount of those errors are expected so long as $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is positive.

5. THE NEW METHOD

In order to gain a clear plan for the final method, the system of the bonds is first explored by means of an expansion of u in powers of y by insertion in the system (1), (2). Suppose that

$$u = a_1 y + \frac{a_2 y^2}{2!} + \ldots + \frac{a_k y^k}{k!} + \ldots$$

where $a_1 \ldots a_k$ represent functions of x. Differentiations with respect to x being denoted by dashes (a_1', etc.); $\frac{1}{\rho}\frac{dp}{dx} = \nu\, f(x)$. It affords in order the following interrelations, if each a_n with $n > 1$, wherever it occurs on the right-hand side, is replaced by its values given several lines above it:

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*The asymptotic formula for $J_{\pm 1/3}$ gives a disappearance constant $\alpha = \frac{\nu}{ay(\lambda/2\pi)^2}$ and a wave length λ proportional to $\frac{1}{\sqrt{y}}$.

a_1 (free), $a_2 = f$ (given), $a_3 = 0$ (given),

$a_4 = \frac{1}{\nu} a_1 a_1'$ (free), $a_5 = \frac{2}{\nu} a_1 f'$ (given × a), $a_6 = \frac{2}{\nu} ff'$(given),

$a_7 = \frac{1}{\nu^2}(4\ a_1^2\ a_1'' + a_1\ a_1'^2)$.(free)

If the series is stopped at this point, the three free values a_1, a_4, a_7 can be obtained from three linear equations, by proceeding from the u values related to three fixed values of y. The first to the seventh powers of the three y values are computed beforehand and used again for every step. Posting a_1 in a_4 gives a_1', a_1 and a_1' in a_7 a_1''. With those values

$a_4' = \frac{1}{\nu}(a_1 a_1'' + a_1'^2)$ can still be computed.

The whole method can be refined by subsequent computation of the terms of eighth and ninth powers with

$$a_8 = \frac{1}{\nu^2}\left[10\ a_1^2\ f'' - 13\ a_1 a_1'\ f' + 9(a_1 a_1'' + a_1'^2)\ f\right]$$

and

$$a_9 = \frac{1}{\nu^2}\left[40\ a_1 ff'' - 16\ a_1 f'^2\right]$$

for the three y values, and then by calculating again a_1, a_4, and a_7 from the given three u values (to be repeated, if necessary, by iteration). The most appropriate choice of the three y values is subject to a special study. It will likely be found appropriate to proceed by first obtaining a first approximation of a_1 from an ordinate y_1; then of a_4 by means of y_1 and y , and an improvement of a_1, lastly a_7 with three y values, an improvement of a_4 and a second improvement of a_1. By this method the second step can be extended to a_8, the third to a_9, the preceding approximation of a_1 and a_4, respectively, being each time posted in the higher terms.

Nondimensional coordinates may be chosen for the numerical calculations, by which, in place of ν the unity ($X = x/l$,

$$Y = y\sqrt{u_0/\nu\, l}, \quad U = u/u_0)$$

substitutes.

This method does not furnish a_7'; hence it cannot be continued without contact with another method which supplies the development of the velocity profile on the inside of the fluid. For it, however, the rule given in equation (4) or else that of equation (5), which probably is even more suitable, is available. This merely needs to be completed by a transition between y and ψ.

Because of

$$u = \sqrt{\frac{2}{\rho}(g - p)} = \frac{\partial\psi}{\partial y}$$

we have

$$y = \Phi(x) + \int_{\psi_1}^{\psi} d\,\psi/u$$

Posting herein

$$g = g(x_0) + \Delta x\left(\frac{\partial g}{\partial x}\right)_{x=x_0} = g(x_0) + \Delta x\;\nu\left(u\,\frac{\partial^2 g}{\partial\psi^2}\right)_{x=x_0}$$

to be improved, if necessary, analogous to the Runge-Kutta method), the y value related to every ψ or u is known up to the unknown value $\Phi(x_0 + \Delta x)$. This denotes a parallel shift of the velocity profile obtained and can be so determined that the profile continuously joins the wall-adjacent profile obtained by the other method. Provision for a wide overlap of both profiles must be made, which thus affords the means for the determination of the coefficient a_7 for the next step. Conversely, the connection for the continuation on the inside is also important, since the numerical determination of $\frac{\partial^2 g}{\partial\psi^2}$ on the edge of the region would each time be accompanied by a loss in width unless the course of g in the edge strip is included.

Naturally, the step of $x_0 + \Delta x$ can be executed also afterward in the wall-adjacent strip backward toward x_0 according to the old method and a correction of the second order derived from the deviations (if necessary also four steps back and forth, so as to lower the error of the step to the fifth order, as in Runge-Kutta's method, which then naturally enables the choice of greater steps Δx).

The method, even as described, is not exactly simple, but it should give more reliable results. Its practical

approval is held in abeyance pending the findings of the Kaiser Wilhelm Institute for Flow Research. It is expected that the continuation can be carried on to the break-away point and probably a little beyond. A study of the conditions on the break-away point $(a_1 = 0)$ shows that

a_1' assumes the form $\frac{0}{0}$, since a_4 along with a_1 becomes zero. But continuation is possible if a_1' and a_4' which here becomes $= a_1'^2/\nu$, is extrapolated as far as that place. From the break-away point $x = x_A$, it probably will be possible to penetrate a little farther with an expansion in powers of $x - x_A$ and y, the coefficients of which are given by the continuous connection with the region $x < x_A$, where, by that time the course of $f(x)$ will have to be chosen on the premise that a dead-air region shall occur.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

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